

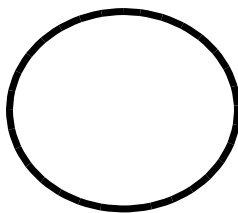
Visualizing mathematical knots using KnotPlot
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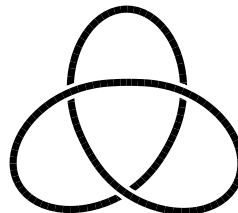
Introduction

KnotPlot is a computer program for visualizing and experimenting with *mathematical knots*. These are slightly different than everyday knots in that they are *closed loops*. This restriction makes the mathematics simpler, and also prevents the knots from being untied (they are considered to be infinitely flexible strings). Knots have been studied from a mathematical perspective roughly from the time of Gauss (in 1833) who applied them to a problem in electrodynamics, but it would not be until the end of the nineteenth century that theory of knots really got underway.

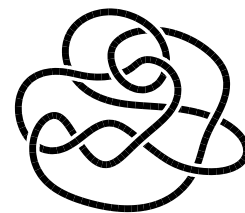
What makes a knot knotted? First of all, we have the simplest knot, the *unknot* shown in Figure 1. The figure on the left shows what looks like a closed loop of string, much like an elastic band. The figure on the right looks a bit more complicated at first, but



Simple unknot



Twisted-up unknot

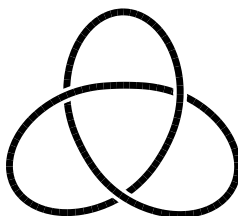


Complicated unknot

Figure 1: Three versions of the unknot.

if you form a string into this configuration, you'll see that by picking it up and "flicking" it, you can get it to look like the figure on the left. So this is what makes a knot not knotted: you can smoothly deform it *without cutting the string* into an ordinary circle. If you don't have a piece of string handy, you can use KnotPlot to convince yourself that the "knot" on the right in Figure 1 is really an unknot.

Now we come to some examples of non-trivial knots. The two simplest are the *Trefoil* and the *Figure-8* knot shown in Figure 2. It's probably pretty clear that neither of these knots can be converted into the unknot without cutting the string or having the



Trefoil

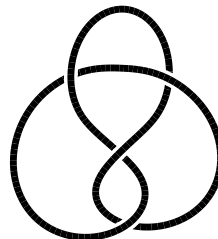
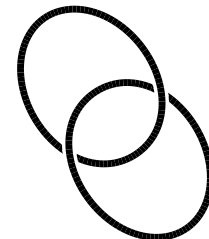


Figure-8



Hopf link

Figure 2: The two simplest non-trivial knots, and the simplest link.

string pass through itself. This fact can be proven rigorously by methods developed by knot theorists. It can also be proven that the two knots shown in Figure 2 cannot be deformed into each other. This is a generalization of the idea of knottedness: two knots are *inequivalent* if one cannot be deformed into the other (without self-intersections or cutting). Also shown in Figure 2 is the simplest non-trivial link (a link is just a knot with more than one string).

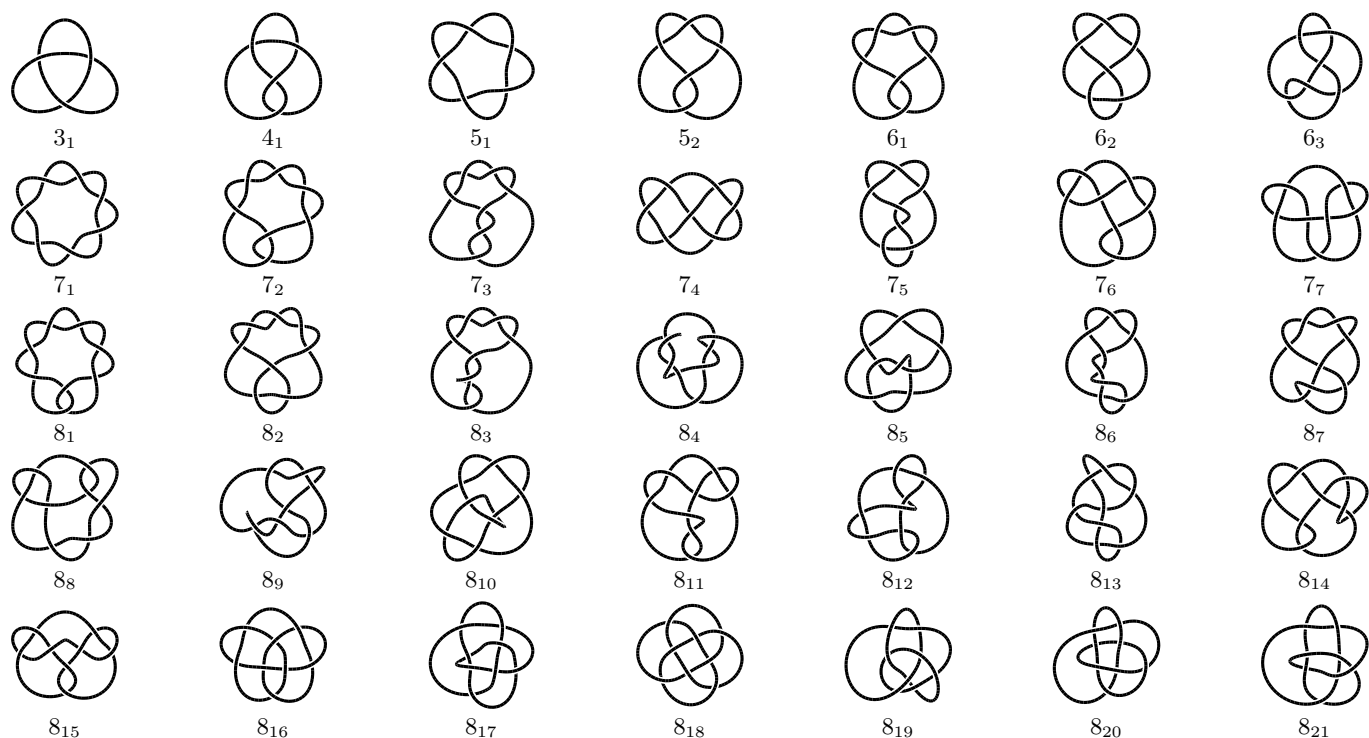


Figure 3: All prime knots with eight or fewer crossings.

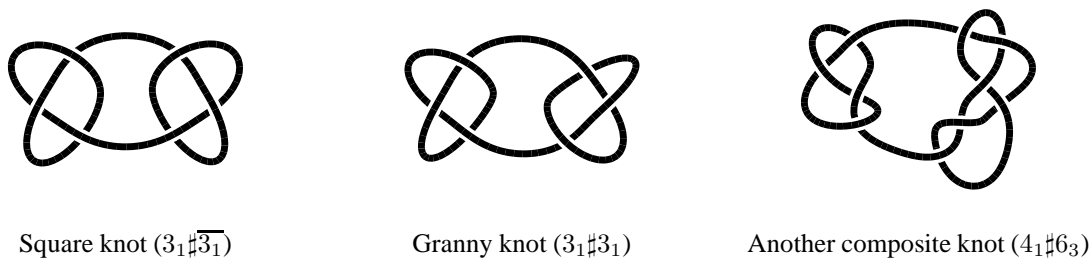


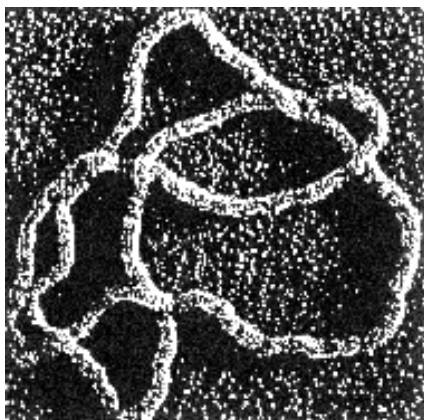
Figure 4: Some composite knots.

These ideas of knottedness allow mathematicians to classify knots in order of increasing complexity. One measure of complexity often used is the *crossing number*, or the number of double points in the simplest planar projection of the knot. There is only one knot with crossing number three (ignoring mirror reflections), the trefoil or cloverleaf knot. The Figure-8 knot is the only knot with a crossing number of four. There are two knots with a crossing number of five, three with a crossing number of six, and seven knots with a crossing number of seven. From there on the numbers increase dramatically. There are 12,965 knots with 13 or fewer crossings and 1,701,935 with 16 or fewer (all in a minimal projection). Figure 3 shows the 35 simplest knots according to crossing number. Knots such as the square knot are usually excluded from knot tables because they can be constructed of simpler knots. Knots that cannot be split into two or more simpler knots are called *prime*, and others such as the square knot and the granny knot shown in Figure 4 are known as *composite*.

Applications

DNA molecules can be knotted (Figure 5a) so it is important to understand the rôle of knotting in the behaviour of these molecules. Knot theory is also being applied to fields as diverse as statistical mechanics, chemistry and quantum gravity, plus knots help keep our shoes on our feet! What more could you want?

However, knot theory stands well on its own as a beautiful part of modern mathematics. In this aspect, knot theory has been elaborated as a wonderful abstraction, including knots in higher dimensions (four, five, or as high as you want, see Figure 5b).



(a) Knotted DNA



(b) Knot in four dimensions

Figure 5: Practical and impractical knots, but equally fun.

For more information

One of the best reference books is *The Knot Book* by Colin Adams (Freeman, 1994). I recommend this book to anyone learning about mathematical knot theory for the first time. It assumes only a general background in mathematics yet contains a great deal to occupy even the expert. Also it has chapters on the recent applications of knot theory to other fields such as physics, chemistry and biology. It even has knot jokes and pastimes. Another good introductory book is *Knot Theory* by Charles Livingston (MAA Press, 1993), although it is more difficult to find than *The Knot Book*. Louis Kauffman's *Knots and Physics* (World Scientific, 1991) is a remarkable book, full of many fascinating ideas (and a lot more mathematics than the previous two books).

The KnotPlot Site on the World Wide Web site is a visual exploration of mathematical knots and links. There you will find hundreds of images (some not so mathematical) along with MPEG movies, and various downloadable goodies. The address is

www.cs.ubc.ca/nest/imager/contributions/scharein/KnotPlot.html

Using KnotPlot for visualizing knots

The simplest thing to do with KnotPlot is just to load up some knots and spin them around! After starting the program, you should see three windows: the main view window, a control panel, and a text window. At the top of the control panel (see Figure 6) are some buttons that you can click to load some knots at random. Another way to load a knot is to use the “Zoo” button. When you have a knot loaded, you can rotate it using the left mouse in the view window (see Figure 7). Imagine that there is a “virtual trackball” surrounding the knot. To scale the scene, use the right mouse (in a left to right motion).

To load a specific knot from the knot catalogue, use the `load` command.¹ For example, type in `load 3.1` to load the trefoil knot (try loading a few knots with the buttons labelled 1, 2, 3, and 4 to see how the naming scheme works). A few “special” knots, such as the Granny and Square knot can be loaded using `load granny` or `load square`, respectively. Try the command `zoo 1` to see a knot zoo similar to Figure 3 or `zoo 252` to see the beginning of the two-component links or `zoo 344` for the three-component links. Typing in `zoo` with no argument or clicking the “Zoo” button will give you a random selection from the catalogue. After you load a knot zoo, click on any knot in the zoo to load that knot.

Now that you’ve started playing with KnotPlot, you can try one or more of the other exercises to learn more about knot theory. I hope you have fun! If you want to get a copy of KnotPlot for yourself, try going to the PIMS web site www.pims.math.ca and follow the “Techno Math” link to the KnotPlot download site. The program is available for Windows 9x/NT, Solaris, Mac OS, Linux, and IRIX. KnotPlot development was generously supported by the Department of Computer Science and the Media and Graphics Interdisciplinary Centre (MAGIC) at U.B.C and by PIMS. I would also like to thank my supervisory committee: Kelly Booth, Jim Little, Dale Rolfsen and Nick Pippenger, for inspiration and helping me get to the end of a turbulent and seemingly endless graduate career.

¹Commands may be typed in with the mouse focus in any of the three windows.

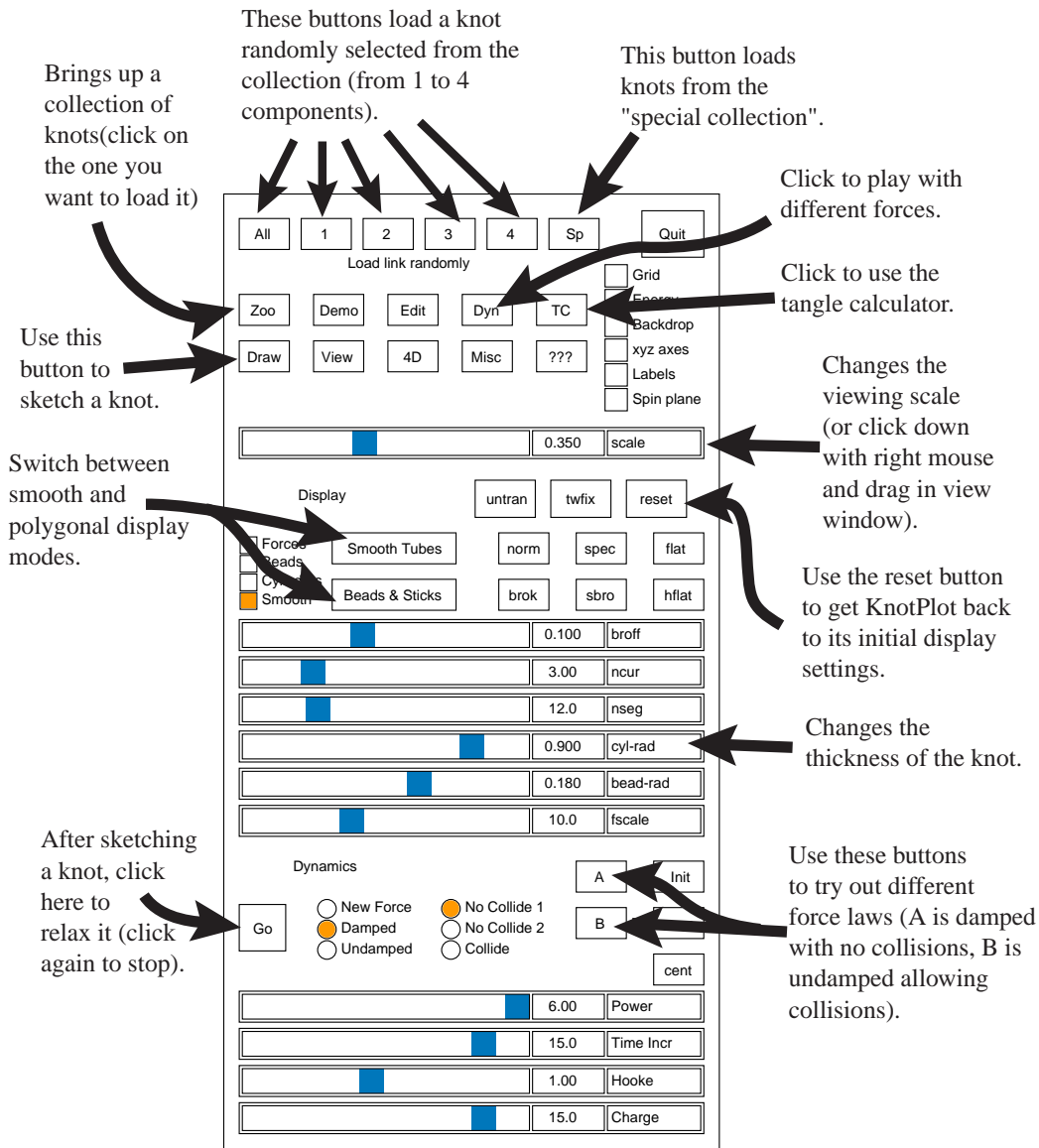


Figure 6: Main control panel, indicating the most useful functions.

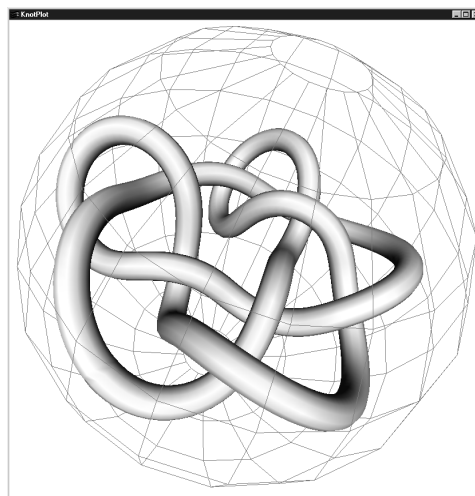


Figure 7: View window showing virtual trackball. Click and drag with the left mouse button to rotate the scene. Click and drag with the right mouse button to scale the scene (move left to right).