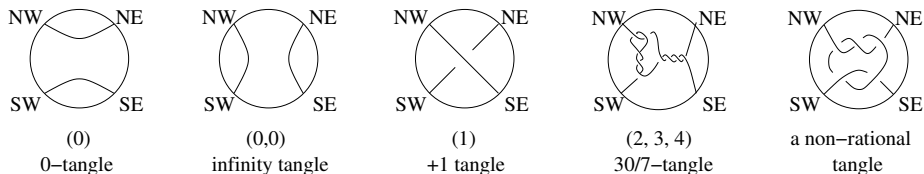


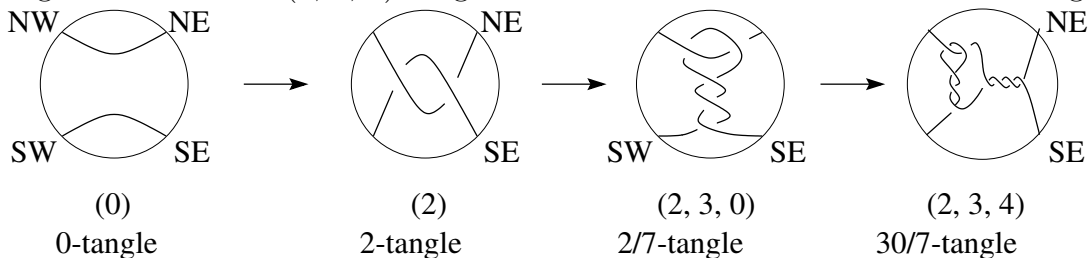
A rational tangle primer

Isabel K. Darcy, idarcy@math.uiowa.edu

A 2-string tangle is a 3-dimensional ball containing two strings and a finite number of circles. Some examples:

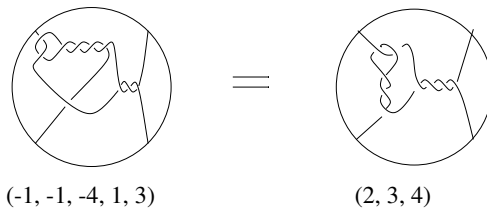


A tangle is rational if it can be obtained from the 0-tangle by alternating between rotating the NE and SE endpoints of the tangle and rotating the SW and SE endpoints of the tangle. For example, the tangle $(2, 3, 4)$ is obtained from the 0-tangle by rotating NE and SE $2 \times 180^\circ$, followed by rotating SW and SE $3 \times 180^\circ$, and then rotating NE and SE $4 \times 180^\circ$. By convention, the tangle corresponding to (c_1, \dots, c_n) always ends with horizontal crossings. Thus the tangle with two horizontal crossings followed by three vertical crossings in the figure below is the $(2, 3, 0)$ tangle since it ends with zero horizontal crossings.

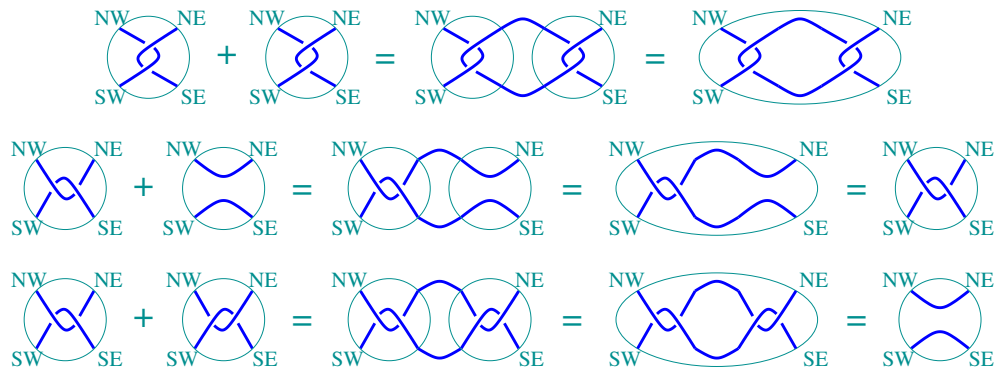


Rational tangles are uniquely identified by their corresponding continued fractions. For example, the tangle $(2, 3, 4)$ is equivalent to the tangle $(-1, -1, -4, 1, 3)$ since

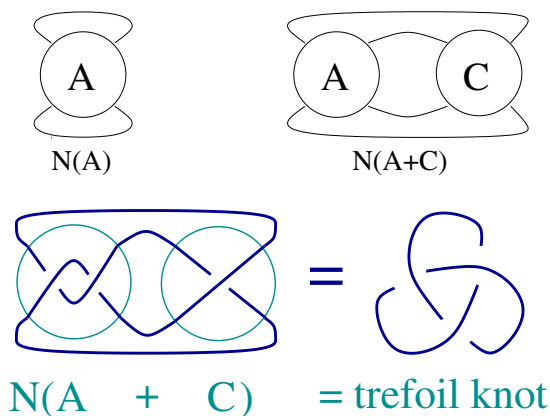
$$4 + \frac{1}{3 + \frac{1}{2}} = \frac{30}{7} = 3 + \frac{1}{1 + \frac{1}{-4 + \frac{1}{-1 + \frac{1}{-1}}}}$$



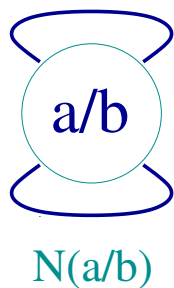
Two tangles can be added by connecting the NW endpoint of the first tangle to the NE endpoint of the second tangle and the SW endpoint of the first tangle to the SE endpoint of the second tangle. For example:



Knots and links can be formed by taking the numerator closure of a tangle or a sum of tangles:



Definition: The numerator closure of a rational tangle is a rational knot or link (also called 4-plat or 2-bridge knot/link).



Two rational knot/links are equivalent if and only if the following relationship holds.

Take $a, c \geq 0$.

$$\begin{array}{ccc} \text{Diagram of } a/b & = & \text{Diagram of } c/d \\ N(a/b) & & N(c/d) \end{array}$$

if and only if

$$a = c$$

and

$$bd^{\pm 1} = 1 \pmod a$$

For example, $N((2, 3, 0)) = N(0 + \frac{1}{3+\frac{1}{2}}) = N(\frac{2}{7})$ and $N((2)) = N(\frac{2}{1}) = N((2))$

Thus, $N((2, 3, 0)) = N(\frac{2}{7}) = N(\frac{2}{1}) = N((2))$ since $7 = 1 + 2(3)$:

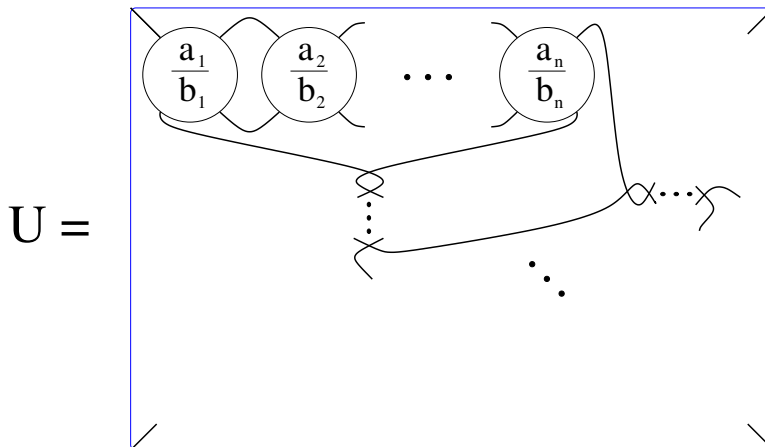
TopoICE-R solves the following system of tangle equations. The user inputs the knots $N(a/b)$ and $N(z/v)$ as well as the tangle $f1/g1$. TopoICE-R solves for the tangles U and $f2/g2$.

$$\text{Diagram of } U \text{ and } f1/g1 = \text{Diagram of } a/b$$

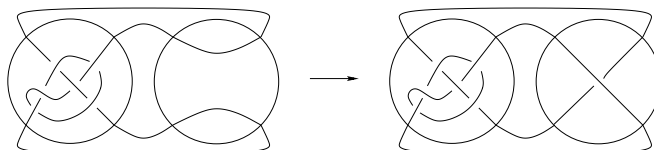
and

$$\text{Diagram of } U \text{ and } f2/g2 = \text{Diagram of } z/v$$

TopoICE-R assumes U is of the following form.



Since $N(a/b)$ and $N(z/v)$ are rational knots, $n = 1, 2$. If $|f_1g_2 - f_2g_1| > 1$, then U must be of this form. Hence TopoICE-R finds all solutions in this case. When $|f_1g_2 - f_2g_1| = 1$, TopoICE-R may not find all solutions since U may not be of this form. For example if the rational knot $N(11/4)$ and the rational link $N(2/1)$ is entered in TopoICE-R and if f_1/g_1 is chosen to be $0/1$, then TopoICE-R misses the following solution where $f_2/g_2 = 1/1$ (observe $f_1g_2 - f_2g_1 = (0)(1) - (1)(1) = -1$) and U is not isotopic to a sum of rational tangles.



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